THE
MATHEMATICAL GAZETTE.

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FUROR ARITHMETICUS.

THOUGH some astronomical observations require ten significant figures for their expression, few observational or experimental results are correct to the sixth figure. Hence in practical questions there is a definite limit, beyond which it is useless to extend arithmetical results.

Mathematical formulæ on the other hand may give results so large as to require expression by a great number of figures, or the result may be incommensurable and more nearly expressed the further the calculation is carried.

Thus, factorial forty is expressed by forty-eight figures

$$\underline{|40}=815915,283247,897734,344051,369596,115894,272000,000000,$$

and the sum of the infinite series e can be found to any number of decimal places.

$$e=1+\frac{1}{1}+\frac{1}{1\,.\,2}+\frac{1}{1\,.\,2\,.\,3}+$$
$$=2{\cdot}718281,828459,045235,360287, \quad .$$

From Plato downwards mystical ideas have been connected with numbers, which have led to the laborious investigation of relations between them. Perfect numbers such as 6 and 28 were found, each of which is equal to the sum of its divisors, and amicable numbers, such as 220 and 284, each of which is equal to the sum of the divisors of the other.

Many curious facts and fancies are collected in the *Numerorum Mysteria* of Peter Bungus (1585).

Owing to the extremely cumbrous notations employed by all the nations of antiquity, except the Hindoos, such calculations could not be carried very far until the introduction of the so-called Arabian notation, which spread over Europe during the thirteenth century. Since that time, though most people have a strong repugnance to undertaking any long numerical calculation, some few seem to be seized with a divine afflatus which carries them through appalling series of figures. Thus, tables are calculated to a great number of places, long incommensurable roots are found, and constants are determined to many figures.

As de Morgan points out (*Budget of Paradoxes*, 290), "These tremendous stretches of calculation—at least we so call them in our day—are useful in several respects; they prove more than the capacity of this or that computer for laborious accuracy, they show that there is in the community an increase of skill and courage."

It is also impossible to know on which side of the line of utility to put any given result. What is useless to-day may in the progress of knowledge become invaluable to-morrow.

When Euler investigated, and Legendre calculated, the Γ function $\int_0^\infty e^{-x}x^{n-1}dx$ from 0 to 1 by thousandths to twelve places, they hardly anticipated that they were enabling Gilbert to obtain the value of Fresnel's integrals $\int_0^s \sin\frac{\pi x}{2}dx$ and $\int_0^s \cos\frac{\pi x}{2}dx$, indispensable in the theory of the diffraction of light.

The majority of mathematicians object to the drudgery of arithmetic. Lord Lytton remarks, even of Newton, "Qui genus humanum ingenio superavit," "That great master of calculations the most abstruse could not accurately cast up a sum in addition. Nothing brought him to the end of his majestic tether like dot and carry one."

A few even of the greatest mathematicians, such as James Bernoulli, Euler, Legendre, Gauss, de Morgan, Hamilton, and Adams have also been expert and laborious computers. Thus Hamilton and de Morgan congratulated one another on finding

$$\cos\frac{2\pi}{7}=0\cdot62348{,}98018{,}58733{,}53052{,}50,$$

to obtain which by the ordinary method they must have summed $\left(x=\frac{2\pi}{7}\right)$

$$1-\frac{x^2}{1\,.\,2}+\frac{x^4}{1\,.\,2\,.\,3\,.\,4}-\quad+\frac{x^{24}}{\underline{|24}}.$$

In several noticeable instances the furor seems to have fallen upon military officers such as Wolfram, Vega, and Oakes, but not upon their naval brethren. It has fallen upon men in all ranks and conditions of life, even upon the great classic Porson, author of the equation

$$xy+zu=444 \qquad xz+yu=180$$
$$xu+yz=156 \qquad xyzu=5184.$$

It has increased the long list of circle-squarers and puzzle makers, and is not unknown among schoolboys. Though the miracles of calculation produced during last century surpassed all previous efforts, they were by no means so widely distributed. The victims of the furor were fewer.

This curious fact seems due to various causes. The commercial production of arithmometers has rendered many extensive tables unnecessary. The modern tendency is towards haste, hence more attention is paid to approximation in the methods of working and in the expression of results. Owing to the spread of educational facilities many, who in former times would have remained on the threshold, become mathematicians and devote themselves to the higher instead of to the lower branches. A still greater number now devote themselves to science and regard calculation merely as a means of expressing the results of experiments, which owing to errors require comparatively few figures.

Thus, in a modern physical research $x^4+44x^3-340x^2-6000x+18000=0$ required solution (*P. Phys. Soc.* xix. 625). The graphical method gave 2·7 as the root with quite sufficient accuracy; in former times a more elaborate and laborious solution would have given $x=2{\cdot}735419339$.

For these and possibly other reasons it seems very doubtful if more powerful elementary tables will be calculated. Such modern tables are merely shortened reprints of results obtained long ago, and attention is chiefly devoted to greater accuracy and convenience in arrangement.

In the higher and newer parts of mathematics, tables such as those of Dr. Meissel and the British Association frequently appear, but they are too often

hidden in the pages of periodicals, and many more seem to be required to deal with recent advances in physics.

Notwithstanding this apparent trend towards approximation in recent times, long series of figures are occasionally required, and the means of ascertaining what results have been already obtained are very inadequate.

Many valuable results are hidden in the pages of old or out of the way periodicals, or in MS. in private libraries, or the archives of learned societies. A few tables and constants to many figures may occasionally be picked up in old book shops, but the lists of publishers will be searched for them in vain. A classified list of such tables and results, both in print and MS., with the places where they can be consulted, would save many an irritating and too often an unavailing search.

If the natural logarithm of π be required, how many remember that it is given by Callet to forty-eight places?

Nat. log $\pi = 1{\cdot}14472,98858,49400,17414,34273,51353,05871,16472,94812,916$.

A brief notice of some of the larger elementary tables may recall what our predecessors have done for us and prove of use.

Crelle (1875) multiples to 1000×1000.

Oakes (1865) reciprocals 1-99999 to seven figures with differences.

Kulik (1848) squares and cubes to 100 000.

Blater (1887) quarter squares to 200 000.

Rheticus, the natural sines for every $1''$ for the first two degrees and for every $10''$ later to 15 places. Some values were added by Pitiscus (1613). Rheticus also calculated a complete $10''$ canon to ten places, published by Otho (1596). According to de Morgan this was "The most laborious work of calculation that any one man ever undertook." Probably no instrument in use at the time was accurate to $1'$.

The discovery of logarithms by Napier in 1614 opened up fresh fields to computers.

In 1624 Briggs published the logarithms from 1-20 000 and 90 000-100 000 to fourteen places. Four years later Vlacq calculated the missing 70 000 logarithms and published the complete table to ten places.

At his death in 1631 Briggs had nearly finished a complete canon to $0{\cdot}01°$, natural sines to fifteen places, logarithmic sines to fourteen places, natural and logarithmic tangents and secants to ten places. This table was published by Gellibrand in 1633.

In the same year Vlacq published a complete $10''$ canon to ten places. These two tables of Vlacq's are the basis of almost all modern tables. Had Briggs' table been adopted we should now be using a much more convenient division of the quadrant.

In 1717 Sharp published the logarithms to 100 and primes to 1100 to sixty-one places, and in 1742 Gardiner printed the logarithms to 1000 and of odd numbers to 1161 to twenty places. These tables were reprinted with additions by Hutton (1785) and Callet (1783), who added a twenty figure table of natural logarithms.

Dodson (1742) published antilogarithms to eleven places corresponding to five-figure logarithms.

Wolfram (1778) natural logarithms to 2200 and 1240 beyond to forty-eight places.

The enormous Tables de Cadastre are still in MS. at the Paris Observatory, but an abstract to eight places has been printed.

In 1871 Sang published a table containing the logarithms 100 000-200 000 from his own calculations. He also left MS. containing logarithms to 20 000 to twenty-eight places, and logarithms 100 000 to 370 000 to fifteen places; also a table of sines and tangents for grades, etc. The MSS. are in the care of the Royal Society of Edinburgh.

In 1876 Peter Gray published a table of radices, by the aid of which logarithms to twenty-four places can be dealt with.

Degen (1824) calculated the logarithms of factorials up to $\underline{|1200}$ to eighteen places.

In 1827 Legendre published a double entry table giving the values of elliptic integrals of the first and second kind for each degree of the modulus and amplitude to ten places.

It is hardly possible now-a-days to realize the enthusiasm and patience of the earlier tabulators, who plunged into an ocean of calculation often with very inefficient formulæ and no modern appliances. Even to use a table to twenty places taxes the industry and accuracy of a modern computer. In general twenty figures must be multiplied by two or three, fifteen figures by fifteen, ten by ten, and five by five. The tedium and risk of error in using such tables is shown by the fact that Callet and Hutton give identical examples. Few will take the further step of using Sharp's table, which has not been reprinted for many years.

Questions connected with prime numbers have taxed the power of mathematicians and the patience of computers.

The only known method of finding primes, or numbers which are exactly divisible only by one, is the tedious "sieve of Eratosthenes" of Cyrene (275-194 B.C.). But Fermat seems to have been in possession of a more rapid method. When asked by Mersenne, in 1643, if 100 895, 598 169 was prime, he replied, it is the product of $898\,423 \times 112\,303$. Rouse Ball remarks "Even with the aid of modern tables I do not think anyone would now undertake to answer such a question."

Tables of primes and factors up to 9 000 000 have been printed by Chernac, Burckhardt, Glaisher, and Dase, and that for the tenth million is said to exist in MS. at Berlin.

The highest prime now known is $2^{61}-1$ or 2,305843,009213,693951.

A convenient method for finding the factors of a comparatively small number consists in finding a square such that when the given number is subtracted from it a square is left. The sum and difference of the two square roots are the factors of the given number (*Nature*, Feb. 28, 1889).

Stanley Jevons asks (*Prin. of Sci.*, 123), "Can the reader say what two numbers multiplied together will produce 8 616 460 799? I think it unlikely that any one but myself will ever know, for they are two large prime numbers, and can only be rediscovered by trying in succession a long series of prime divisors until the right one is fallen upon. The work would probably occupy a good computer for many weeks."

It is proverbially unsafe to prophecy before the event. Subtracting from the next higher square,

$$\begin{array}{r l} 8\,616\,480\,625 & \text{or } 92825^2 \\ \underline{8\,616\,460\,799} & \\ 19\,826 & \end{array}$$

Since $(a+n)^2 = a^2 + 2an + n^2$, if any number of the form $n(2a+n)$ be added the sum remains a square. Hence, a number of the form $n(185\,650+n)$ must be added to 19826 such as to make the sum a square. The number of trials may be reduced by remembering the criteria of squared numbers, such as that they must terminate in 00, 1, 4, 5, 6 or 9. In the present case n is found to be 55. $92825+55=92880$.

$$\begin{array}{r r} 55(185\,650+55)= & 10\,213\,775 \\ & \underline{19\,826} \\ & 10\,233\,601 = 3199^2. \end{array}$$

$$\begin{array}{r r} 92\,880 & 92\,880 \\ \underline{3\,199} & \underline{3\,199} \\ 96\,079 \times & 89\,681 = 8\,616\,460\,799. \end{array}$$

It is obvious that for numbers so large as to be beyond the reach of an available table of squares the method may require much patience. It is easier the more nearly equal the factors are. Luckily the subject of primes, however important in the theory of numbers, does not enter much into practical calculations.

In solving a long or difficult equation it is generally worth while to draw the graph on squared paper, which gives the first figure or two of the roots, and then to consider if there is any way of dodging the difficulties.

It looks a little formidable at first to find the four roots of the equation

$$70x^4 - 140x^3 + 90x^2 - 20x + 1 = 0$$

until it is noticed that the sum of the four roots is 2, and that if x be replaced by $t + \frac{1}{2}$, the equation reduces to

$$70t^4 - 15t^2 + \tfrac{3}{8} = 0,$$

a quadratic in t^2, and the roots are

$$\begin{aligned} x &= 0{\cdot}06943,18442,02975,4 \\ &= 0{\cdot}93056,81557,97024,6 \\ &= 0{\cdot}33000,94782,07567,7 \\ &= 0{\cdot}66999,05217,92432,3. \end{aligned}$$

For practical purposes it is, in general, only required to obtain one real root of an equation or number. The most generally accepted method is that of Horner, which is given in the text-books.

Find a number such that the sum of the first five powers is equal to 100.

$$x^5 + x^4 + x^3 + x^2 + x - 100 = 0. \quad x = 2{\cdot}239\,643.$$

Not only was de Morgan a victim of the furor, but he infected his pupils. Hicks found the real positive root of the classical equation

$$x^3 - 2x = 5,$$

$x = 2{\cdot}09455$ to 152 places.

Valuable as Horner's method is, in some cases the far less well-known method of Weddle seems to give the result with considerably less labour and risk of error.

Weddle gives the two following solutions which it would be very tedious to find by Horner:

$$(100\,000)^{\frac{1}{11}} = 2{\cdot}848\,035\,869.$$

$$1379{\cdot}664x^{622} + 2686034 \times 10^{432}x^{153} - 17290224 \times 10^{518}x^{60} + 2524156 \times 10^{574} = 0.$$

$$x = 8{\cdot}367\,975\,431.$$

Hutton's method of approximating to the root of a number seems not to be so generally known as it deserves to be. If a be nearly the n^{th} root of N, a still nearer root is

$$\frac{\overline{n+1}N + \overline{n-1}a^n}{\overline{n-1}N + \overline{n+1}a^n} \times a.$$

In the case of the cube root this reduces to

$$\frac{2N + a^3}{N + 2a^3} \times a.$$

If $N = 10$ and $a = 2{\cdot}154$ the next approximation gives

$$\sqrt[3]{10} = 2{\cdot}154\,434\,690\,02.$$

It may encourage the would-be computer to mention a few results which have been already obtained.

In 1657 Gaspar Schott found 2^{256}, which consists of 78 figures.

In 1863 Suffield and Lunn calculated the recurring period of 1/7699 consisting of 7698 figures, but this was beaten by Shanks, who found 1/17 389 to 17 388 figures.

Colville, a pupil of de Morgan, found $\sqrt{2}$ to 110 places. A preliminary attack may be made on $\sqrt[3]{19}=2{\cdot}668\,401\,648\,721\,944\,867\,339\,630$.

Probably no constant has caused the expenditure of so much labour as π. Illegitimate attempts to obtain an exact value are immortalized in the *Budget of Paradoxes*. Legitimate approximations have culminated in Shanks' value to 707 places. He also obtained e, $\log_e 2$, 3, 5, 10 and M to 205 places.

Adams, the discoverer of Neptune, was also an ardent computer. He found the numbers of Bernoulli from B_{32}–B_{62}, and more roughly to B_{100}. He also calculated Euler's constant E, $\log_e 2$, 3, 5, 7, 10 and M to 270 places, and the sums of the reciprocals of the first 500 and the first 1000 natural numbers to 260 figures.

The sum of the tenth powers of the first thousand natural numbers is

$$91{,}409{,}924{,}241{,}424{,}243{,}424{,}241{,}924{,}242{,}500.$$

James Bernoulli mentions that it took him rather less than seven and a half minutes to obtain this result.

Though, owing to unavoidable errors, it is useless to carry the arithmetical reduction of observational or experimental results beyond a comparatively small number of figures, the reduction, especially in astronomy, may severely tax the power and patience of the computer.

The value of the average result of equally good observations increases as the square root of their number. Hence, if a quantity is to be determined as accurately as possible, a large number of observations or experiments must be made, and if the method of least squares be rigorously applied the work becomes very tedious.

Theory shows that the length l of a pendulum in latitude ϕ is determined by the equation

$$l=l_0(1+k\sin^2\phi),$$

where k is a constant to be determined by observation at different places. Bowditch undertook the tremendous labour of combining 52 observational values in strict accordance with the method of least squares to obtain the equation

$$l=39{\cdot}01307+0{\cdot}20644\sin^2\phi.$$

The survey of a country entails an immense amount of arithmetical work in obtaining the most probable values from slightly discrepant observations and in solving a vast number of spherical triangles. A large staff of computers may be occupied for years.

A good many dodges by which long calculations may be lightened may be found in modern arithmetics. Special reference may be made to de Morgan's article on "Computation" in the *Supplement of the Penny Encyclopædia*, to Boccardi's *Guide du Calculateur*, and to Langley's *Computation*.

Many of these are of general utility, but too often a special dodge is required, which only suggests itself when the work is half done.

Some fruits of more or less bitter experience may be jotted down in the hopes of saving a neophyte from some of the numerous pitfalls which beset the way of the computer.

1. When long arithmetical results are required, endeavour to ascertain if they have been already found and lie buried in periodicals, old books, or MSS.

2. Draw a graph representing the data on squared paper, making free use of Prof. Perry's black thread. Or obtain an approximate result with the aid of a slide-rule or four-figure table of logarithms.

3. Obtain a formula as convenient as possible for calculation, having regard to the personal peculiarities of the computer, and to the tables and other aids available.

4. Consider to how many figures the data are accurate, or the answer required, and work to one or two more.

5. Use logarithmic or other tables of only the required accuracy, and correct them from the table of errata or otherwise. Errors are somewhat numerous in many of the older tables.

6. Obtain a quantity of "mark paper," ruled in small squares, and rule each fifth or sixth vertical line in red.

7. Write the nine multiples of numbers, which are frequently required, on slips of card; these slips can be arranged as required on a board by the aid of drawing pins. Blater's Table of Napier or Sawyer's Automatic Multiplier may be used instead of the slips.

8. A few wooden or metal slips are useful for ranging long rows of figures or covering up any not required.

9. It is a counsel of perfection to repeat a tedious calculation from a different formula with different tables.

It is to be remembered that in the value of π, published by Rutherford in 1841, to 208 places, only 152 figures are correct. Two errors crept into Shanks' result to 530 places in 1853. If such computers publish erroneous figures it may well behove their inferiors to be careful. SYDNEY LUPTON.

CORRESPONDENCE.

APPROXIMATION IN METHOD VERSUS APPROXIMATION IN ARITHMETIC.

To the Editor of the *Mathematical Gazette.*

DEAR SIR,—In the welcome Report on the Correlation of Mathematical and Science Teaching by the Joint Committee of the Mathematical Association and the Association of Public Schools' Science Masters, two examples are given (p. 6) on the method which should be followed in treating problems in Physics.

There can be no question of the main contention that great care should be taken that the pupil is not finding a numerical result simply by substitution in a given formula. But the first example as given raises another point also. The example is on the linear expansion of a brass rod, and is begun by directing the pupil's attention to the meaning of the coefficient of linear expansion as "the amount by which unit length (no temperature given) expands when heated through unit temperature." In working the example this unit length is tacitly assumed to be at 10° C.—or else it is tacitly assumed that there will be no appreciable difference in the result whether this unit length be taken to be at 0° C. or at 10° C.

This vagueness in method raises a point of considerable importance in the teaching of such questions when clothed with all the authority of occurring in a specially recommended example in a Report of such weight. But I venture to ask whether it is well to allow unnecessary inaccuracies in method simply for the sake of shortness and saving a little mathematics? I am not here speaking of approximations to what really occurs in Nature which must be assumed sufficiently to simplify Physical problems. But would it not be far better to work the theoretical parts of the problem clearly and logically from the accepted definitions for the Physical quantities (these definitions having probably been explained carefully and at length to the class), and then find the approximate numerical answer by accurate approximate arithmetic? By accurate approximate arithmetic is here meant such that the student knows to which significant figure he can trust. With logarithms or a slide rule this final arithmetic is short and easy, and will not withdraw the student's attention from the main principles of the problem.

A clever boy or girl will not be confused by such a treatment of the problem as that given in the Report, but, in my experience, the average pupil is only confused by such tacit approximations in method. In this particular case of expansion this vague confusion in such a pupil's mind causes trouble when the gaseous laws are considered, viz.: why should the volume of a gas be referred back to 0° C. and not to the temperature of the room? Or difficulties arise in problems where the Fahrenheit scale is used, and so on.

The example is—"A brass rod is 25 metres long at 10° C., find its length at 50° C., if the coefficient of linear expansion of brass is ·000018."

One metre of brass at 0° C. heated 1° C. expands so as to have length

$$1+\cdot 000018 \text{ metres.}$$

One metre at 0° C. heated to 10° C. expands so as to have length

$$1+10\times\cdot 000018 \text{ metres.}$$

∴ one metre at 10° C. if cooled to 0° C. has length

$$1/(1+10\times\cdot 000018) \text{ metres.}$$

∴ 25 metres at 10° C. if cooled to 0° C. has length

$$25/(1+10\times\cdot 000018) \text{ metres.}$$

One metre of brass at 0° C. when heated to 50° expands so as to have length $1+50\times\cdot 000018$ metres.

∴ $25/(1+10\times\cdot 000018)$ metres at 0° when heated to 50° expands so as to have length $25\dfrac{1+50\times\cdot 000018}{1+10\times\cdot 000018}$.

This can be worked out by logarithms, or else continued

$$=25(1+50\times\cdot 000018)(1-10\times\cdot 000018) \text{ nearly} \quad\ldots\ldots\ldots\ldots(a)$$
$$=25(1+40\times\cdot 000018) \text{ nearly} \quad\ldots\ldots\ldots\ldots(b)$$
$$=25\cdot 018,$$

and the degree of approximation at stages (*a*) and (*b*) can be seen at once by any student familiar with elementary approximate methods in Algebra.

This is certainly somewhat longer than as given in the Report, but if our object be to correlate Mathematics and Physics at school, why should we teach our Physics both vaguely and illogically from the given definitions merely in order to avoid giving our boys and girls a little practice in elementary mathematics? Yours, etc.,

EDITH A. STONEY,

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TO THE EDITOR OF THE *Mathematical Gazette.*

SIR,—In a recent issue you threw out a suggestion for a pillory for examination questions. I beg to enter the following:

"The external measurements of a closed box are 36 inches, 2·2 feet, and ·50$\dot{6}$ yards. Find the cubic space within if the wood of which it is made has a uniform thickness of one-tenth of a foot."—Board of Education, 1904.

Note the useful 'it,' the mixture of units, and the recurring decimal. English grammar, ordinary common sense, and physical possibility smashed in one question! Can anyone beat this?

Some obvious and rather painful reflections are suggested by the fact that the question emanates not from an obscure and ill-paid schoolmaster, but from the Board of Education. Yours faithfully, ALEPH.

MATHEMATICAL NOTES.

314. [V. I. a.] *On* $\frac{a}{b}=\frac{c}{d}=\frac{a+c}{b+d}$.

Certain algebraical theorems appear more difficult to some beginners than would have been anticipated beforehand.

Thus the proof of the proposition that if $\frac{a}{b}=\frac{c}{d}$ each ratio is equal to $\frac{a+c}{b+d}$ is apparently much easier than that of the proposition that the sum of the roots of $x^2+px+q=0$ is $-p$.

Yet it will, I think, be agreed that the former proposition is considered the more difficult by many boys and is least thoroughly comprehended or readily applied.

One hardly likes to offer such a suggestion as that which follows, but I am not sure that this is not a mistaken feeling. To put such ideas forward as being novel or original would indeed be foolish, but in teaching children a childish remark may be very much to the point.

Moreover, there are many people teaching mathematics who are very hard worked in term time and who experience the pressure of the holidays rather severely at other times, and so have little time for thinking things over quietly.

The suggestion is that some concrete illustrations of the proposition may render its simplicity manifest and prepare the way for an enunciation of the result in its most general form.

1. A regiment of 1000 men contains 130 Yorkshiremen: another of 1200 men contains 156.

The proportion of Yorkshiremen in each regiment is the same, for

$$\frac{130}{1000}=\frac{156}{1200}.$$

The proportion of Yorkshiremen in the total force is

$$\frac{130+156}{1000+1200};$$

and this is obviously the same as the proportion in either regiment.

2. A boy gets 34 out of 100 for arithmetic, 68 out of 200 for algebra, 51 out of 150 for geometry.

The proportion of marks on each paper is the same, for

$$\frac{34}{100}=\frac{68}{200}=\frac{51}{150};$$

and the proportion of the total marks on all three papers is

$$\frac{34+68+51}{100+200+150};$$

and as the boy got 34 per cent. on each paper, he must have got 34 per cent. on the total.

3. Let the student draw up a statement of a rule which will apply to both these examples. The obvious graphical illustration should of course also be given.

C. S. Jackson.

315. [L^1. 17. e.] *Systems of conics whose director circles have a common radical centre.*

A conic is uniquely determined by five pairs of conjugate lines. Taking two of the pairs to be the isotropic lines through two fixed points, we see that the director circle of a conic having three assigned pairs of conjugate lines is uniquely determined when two of its points are given. Hence the

director circles of conics having three assigned pairs of conjugate lines have a common radical centre, and therefore also a common orthogonal circle which may be real or imaginary.

Two particular cases of this theorem are well known. The director circles of conics having a given self-conjugate triangle, or of conics inscribed in a given triangle, have a common orthogonal circle. The position of the orthogonal circle in either case is readily seen by a consideration of special conics of the system. Thus, in the former case, taking the conic to be a small ellipse having two of the sides of the self-conjugate triangle as conjugate diameters we see that the orthogonal circle is the circumscribing circle of the self-conjugate triangle. In the latter case taking the conic to be a thin ellipse with one end at a vertex and the other on the base of the circumscribing triangle we see that the orthogonal circle must be the self-polar circle of the triangle.

The general theorem is that the director circles of all conics harmonically inscribed in three given conics have a common orthogonal circle.

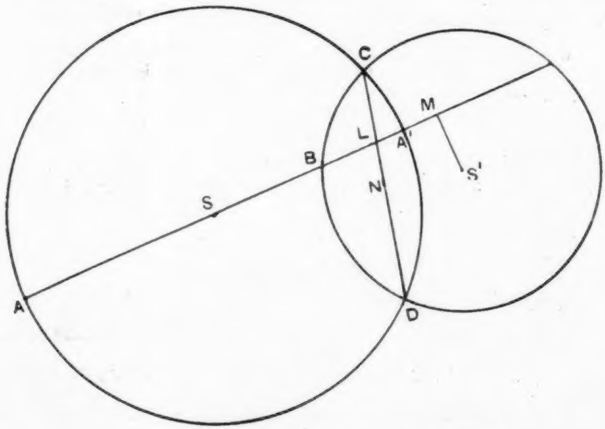
E. J. Nanson.

316. [K. 11. a.] *On orthogonal circles.*

If the diameter AA' of a circle cuts an orthogonal circle in B, B', and cuts the common chord CD in L, then

$$\frac{AL}{LA'}=\left(\frac{AB}{BA'}\right)^2=\left(\frac{AB'}{B'A'}\right)^2.$$

Proof. Let M be the mid-point of BB', and N the mid-point of the common chord, and S, S' the centres of the circles.



Then $$SL.SM=SN.SS'=SC^2;$$

$$\therefore\ (AA',\ LM) \text{ is harmonic}$$

and $$\frac{AL}{LA'}=\frac{AM}{A'M}. \qquad\qquad (1)$$

Also, since (AA', BB') is harmonic, and M bisects BB',

$$\therefore\ AM.A'M=BM^2;$$

$$\therefore \frac{AM}{BM}=\frac{BM}{A'M}=\frac{AM-BM}{BM-A'M}=\frac{AB}{BA'}, \text{ which also} = -\frac{AB'}{B'A'};$$

$$\therefore \frac{AM}{A'M}=\left(\frac{AB}{BA'}\right)=\left(\frac{AB'}{B'A'}\right)^2. \quad \text{..........................(2)}$$

Equations (1) and (2) prove the proposition. A. LODGE.

317. [**A. 1.**] In a recent number of the *Gazette* suggestions were asked for symbols for "is approximately equal to but greater than," and similar phrases.

For this particular phrase I would suggest the sign of equality barred on the side of the smaller quantity, a symbol resembling a croquet hoop.

Thus $a \sqsupset b$

means "a is approximately equal to b but greater than it."

The similar symbols $a \sqsubset b$, $a \mathrel{\mathrm{I}} b$

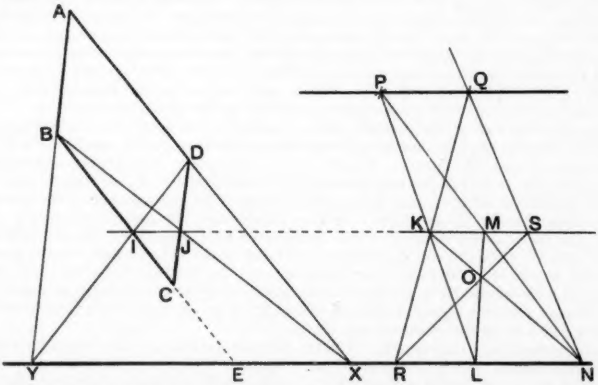
call for no explanation, the latter being used when the relative magnitude is dubious. Thus, if θ be a small positive angle,

$\sin\theta \sqsubset \theta$ and $\tan\theta \sqsupset \theta$. JOHN H. LAWLOR.

318. [**K. 21. a.**] *Note on the problem: "Given a parallelogram, construct a parallel to a given line through a given point."*

In the *Mathematical Gazette* for Oct. 1908, Mr. M. I. Trachtenburg states that solutions based on homology are given in Cremona's *Projective Geometry* and in Mr. J. W. Russell's *Treatise on Elementary Geometry*, and he himself gives a construction based on a ruler construction for the fourth harmonic, which is used five times. Hence, although an extremely elegant construction when considered theoretically, it would nevertheless be almost interminable in practice, and except in the hands of an extremely careful draughtsman almost impossible of execution.

All of which inclines me to think that the following method, depending for its proof on very simple "sixth book" geometry, is new and may be of interest.



Let $ABCD$ be the given parallelogram, YX the given line, and P the given point.

I. **Firstly, construct *a* parallel to YX.**

Produce AB, AD to cut YX in Y and X; join BX, DY cutting DC, BC in J, I; then IJ is parallel to YX.

For $BI : IE = AD : DX = BC : DX = BJ : JX$;

$\therefore\ IJ \parallel EX$.

II. **Secondly, given the *two* parallels IJ, YX, draw a third parallel through P.** Draw PKL, PMN cutting IJ, YX in K, M and L, N; let LM, KN meet in O; draw ROS cutting YX, IJ in R, S; let RK, NS meet in Q: then PQ is parallel to YX.

For $NP : NM = LN : KM = NO : OK = RN : KS = NQ : NS$;

$\therefore\ PQ \parallel MS$ (IJ or YX). J. M. Child.

REVIEWS.

RECENT WORKS ON VECTOR ALGEBRA.

Vector Analysis. By Joseph George Coffin, B.S., Ph.D. John Wiley & Sons, New York; Chapman & Hall, London. 1909.

Elementi di Calcolo Vettoriale. By C. Burali-Forti and R. Marcolongo. Nicola Zanicelli, Bologna. 1909.

Omografie Vettoriali. By C. Burali-Forti and R. Marcolongo. G. B. Petrini di Giovanni Gallizio, Torino. 1909.

Einfuhrung in die Vektoranalysis. Von Dr. Richard Gans. 2nd Edition. B. G. Teubner, Leipzig and Berlin. 1909.

Axiomatische Untersuchungen über die Vektoraddition. Von Rudolf Schimmack. Nova Acta. Abh. der Kaiserl. Leop. Carol. Deutschen Akademie de Naturforscher. Band XC. Nr. 1. Halle. 1908.

A Vector quantity may be combined with another similar quantity according to the parallelogram law which is familiar in the composition of velocities and forces. That may be taken as a definition of a vector quantity. The simplest example of a vector quantity is a displacement in space and this may be represented by a directed line. Suppose we take it so and *define* a vector to be a finite real directed line in Euclidean space. It then becomes a question of fundamental mathematical enquiry, what simplest set of axioms will give the law of Vector Addition. This is the object of Rudolf Shimmack's memoir, which therefore appeals to those mathematicians who are interested in such enquiries. Most of us are quite content to take the law of vector addition for granted as a direct result of physical experience; and push on to the developments of vector analysis, such as we find presented to us in the books named above.

But why will vector analysts always be rushing to the front with new notations? I have carefully read the modern books and articles on vector analysis by Gibbs, Heaviside, Föppl, Macfarlane, Henrici, Gans, Bücherer, Jahnke, Grassmann, and now these last, and—leaving out of account Gibbs's *Dyadic*—I find that they are all, where they treat of the same things, remarkably like one another, although they differ superficially in notation. In one thing they do agree—namely, a total disregard for the real essence of the quaternion vector analysis, from which to a large extent they have, sometimes without acknowledgment, and perhaps without knowledge, drawn their inspiration. In the essential feature which marks these systems off from Hamilton's method, they are all more or less copies of O'Brien's vector algebra, invented about the time during which Hamilton was rapidly developing his powerful calculus.

Coffin presents us simply with Willard Gibbs's notation. He thinks it best for the stated reason that it is symmetrical. But the "vector product $\alpha \times \beta$"* is *not* symmetrical, for it is not equal to $\beta \times \alpha$. The truth is that each author likes the

* I follow Hamilton in using Greek letters for vectors and Roman letters for scalars; for although the black Roman type now so popular for the vector is sufficiently distinctive in the printed page it is not better in that respect than the Greek letter, and cannot compare for a moment with the latter for real manipulative work with pen or pencil.

notation he is accustomed to. In his comparison of notations on p. 223, Coffin gives Hamilton's inaccurately in one place and incompletely in another—thereby proving that his knowledge of it is superficial. His book is intended to be a working book for the student, and lays no claim to logical presentation of the subject—"no attempt at mathematical rigor is made" we read in the preface. But that does not excuse loose definition and vicious reasoning in a circle. The book opens with the erroneous statement that "A vector is any quantity having direction as well as magnitude." On page 40 we read, "In fact all quantities representable by Vectors obey the parallelogram law." It would be more logical to define once for all that Quantities which obey the parallelogram law are Vector quantities. A little later we read, "Since angular velocities may be represented by vectors they should compound according to the parallelogram law." This statement follows a clear account of the "convenient" geometrical representation of an angular velocity by means of a measured length along the axis of rotation. That this directed line is a vector is *proved* by the kinematical truth that angular velocities so represented compound according to the parallelogram law. Coffin's so-called definite proof of the converse assumes the law of composition right through and is indeed no proof in any strict sense. In most respects, however, this book is an admirable presentation of Willard Gibbs's method, giving many examples for the students to work at, and covering a large ground in its physical applications. Gans's work lays most stress on the applications of vector analysis to electro-dynamics. He uses round and square brackets to distinguish scalar and vector products; but in method it is logically identical with Gibbs's system. Personally I regard these systems as inferior to Hamilton's not only in their analytical conception but also in their practical power as a calculus. This inferiority is a feature of all the non-associative vector algebras which have been devised, including that which is now offered to us with all the authority of the two Italian mathematicians, Burali-Forti and Marcolongo.

Burali-Forti and Marcolongo claim that their system is the *minimum* system suitable for a large field of geometrical, physical, and mechanical problems. Instead of Gibbs's $\alpha \,.\, \beta$ and $\alpha \times \beta$ for the "scalar and vector products" they use $\alpha \times \beta$ and $\alpha \wedge \beta$, introducing an inverted $\vee$ as the symbol of the vector product. Of course there is absolutely nothing in this shuffling with symbols. The real virtue of a vector analysis cannot depend upon whether we use a dot or a cross or a wedge or a stroke interposed between the two vectors to represent what is generally called the vector product. The objection I have to all these notations is that they are essentially artificial and not in accordance with the usual methods of analysis. For example, Burali-Forti and Marcolongo follow the usual custom and write ang $(\alpha\beta)$ and sin $(\alpha\beta)$ for the angle between α and β, and the sine of that angle respectively, and yet maintain that the "vector product" is best symbolised by a separating symbol placed between the two constituent vectors. A most important *factor* of this "vector-product" is this very sin$(\alpha\beta)$. The "vector-product" is in fact a function of α and β, and in no true analytical sense a product of them. Thus we cannot pass from the equation $\alpha \wedge \beta = \gamma$ to an equation of the form $\alpha = \gamma \div \beta$.

Why will so many vector analysts refuse to have anything to do with the complete analytical product of vectors? Analytically the vector quantity differs from the ordinary algebraic quantity in not satisfying the commutative law in multiplication. The products $\alpha\beta$ and $\beta\alpha$ are not the same. A geometrical meaning can be assigned to them; and the functions $\alpha\beta + \beta\alpha$ and $\alpha\beta - \beta\alpha$ are found to possess certain important properties, which make them of great use in geometrical and physical investigations. This is practically Hamilton's mode of approach; and his $S\alpha\beta$ and $V\alpha\beta$, the Scalar and the Vector respectively of the product $\alpha\beta$, are immediately deduced in all their significance *without further definitions.*

It is the necessity for the introduction of definition after definition which declares the artificiality of the modern systems of vector analysis. Burali-Forti and Marcolongo have some novel ways of doing this. Having defined their vector product after the usual fashion they define the "internal product" $\alpha \times \beta$ as the real number by which we must multiply any vector ρ normal to β so as to obtain the vector $(\alpha \wedge \rho) \wedge \beta$. Then by introducing a second vector σ also perpendicular to β they deduce the meaning of $\alpha \times \beta$. What advantage has this circuitous definition with its adventitious vectors which in no way affect the value

of $\alpha \times \beta$—what advantage has this over the simple statement, let $\alpha \times \beta = ab \cos \theta$? By the non-quaternionic approach, some definition must be given—why not choose the simplest?

The same tendency to start with obscure and complex definitions is shown in their treatment of the now familiar quantities, the Curl (or *Rotacione*) and the Divergence of a vector—written by them *rot* and *div* respectively. Already they have defined *grad* of a scalar in Tait's way. That is, they define it by the equation $du = (\text{grad}\, u) \times dP$ which is equivalent to Tait's

$$du = -Sd\rho\nabla u.$$

P is used to represent the point in space and dP is its displacement. dP is practically equivalent to the usual $d\rho$, where ρ is the vector drawn from a fixed origin to the variable point P.

Tait makes his definition of ∇ the nucleus of the whole development and deduces *all* the properties of ∇ with its grad, curl, and divergence meanings *without a single other definition.* This is not possible in Willard Gibbs's system, nor is it possible in the exactly similar system expounded by Burali-Forti and Marcolongo. They introduce definition after definition, and yet with all their ingenuity they never get a real single manipulative operator. The simple reason of all this unnecessary complexity in definitions and rules of combination is that they will not work with the complete product $\nabla\sigma$. I have not space to discuss their definitions of $V\nabla\sigma$ and $S\nabla\sigma$—their *rot* σ and $-$*div* σ. It will suffice to write down the quaternion *identity*

$$S\nabla\sigma \equiv -Sa^2\nabla\sigma \equiv -Sa\nabla\sigma a \equiv -Sa(\nabla S\sigma a + \nabla V\sigma a),$$

and to state that in that last form with the adventitious unit vector a Burali-Forti and Marcolongo find their definition of *div* σ, in terms of the *grad* and *rot* of functions of σ and a.

In their book on the "Elements," these advocates of the so-called minimum system use the symbols $\times$, $\wedge$, grad, rot, div, $K\sigma$, Δ_2, Δ'_2, instead of Hamilton and Tait's S, V, ∇; and yet they call their system the minimum system!

The symbol $K\sigma$ introduced above is a linear vector operator. The quantity $K\sigma\xi$ is equivalent to the Hamiltonian expression $-\nabla S\sigma\xi$ where ∇ acts directly only on the σ. To see how the Italian vector analysts treat the linear vector function we must turn to their other book, *Omografie Vettoriale*, which is a very able exposition of Hamilton's Linear Vector Function. Hamilton used ϕ—they use a; Hamilton used m, m', m'', or m_1, m_2, m_3 for the scalar invariants—they use $I_1 a$, $I_2 a$, $I_3 a$; adopting, it should be noted, a Hamiltonian *principle* of notation (the first Invariant of a, etc.) although they object to Hamilton's $Va\beta$ (Vector of the product $a\beta$). Hamilton employed an auxiliary function ψ, and used ϵ for the vector invariant defined by the relation

$$(\phi - \phi')\rho = V\epsilon\rho,$$

where ϕ' is the conjugate of ϕ; they use other auxiliary functions Ra and Ca, with the conjugate Ka (appropriating another of Hamilton's notations) and represent the vector invariant by Va. Outside these superficial changes there is absolutely nothing in their *omografie* to distinguish it from Hamilton's linear vector function. In spite of their claim in the preface no quaternionist could ever admit that this part of their calculus has greater potentiality than Hamilton's. It cannot have—it is exactly the same thing.

It is of great interest, however, to see how Burali-Forti and Marcolongo work out the "derivates" of vector functions—not that there is anything new in their conceptions, but they have developed very skilfully a notation which Hamilton himself noted as a possible though not an analytically convenient mode of representation. Let σ be any vector function of ρ, then $d\sigma$ is a function in every term of which $d\rho$ will occur once and only once. Therefore $d\sigma$ is a linear vector function of $d\rho$. Using the Leibniz notation symbolically, we may write

$$d\sigma = \frac{d\sigma}{d\rho} d\rho,$$

where however $\frac{d\sigma}{d\rho}$ cannot be regarded as the limit of a ratio but is a linear vector function of $d\rho$. The quantity $\frac{d\sigma}{d\rho} a$ will represent the quantity of the same form

as $d\sigma$, but with a substituted for $d\rho$ throughout. In the quaternion vector analysis as developed by Tait this quantity is written

$$d\sigma = -Sd\rho\nabla \cdot \sigma,$$

and hence the linear function $\frac{d\sigma}{d\rho}$ may be represented symbolically as

$$-S(\quad)\nabla \cdot \sigma.$$

This is the natural way of developing the subject in the quaternion analysis; and no further definition of any kind is needed. All evolves along the lines of the calculus, and the distinctly awkward notation $\frac{d\sigma}{d\rho}$ or $\frac{d\sigma}{dP}$ is avoided. But in the "minimum system" there is no ∇; and Burali-Forti and Marcolongo very skilfully contrive to meet the situation by adopting the notation indicated above. They develop many relations, most of which however are quite familiar to the quaternion worker, and indeed hardly call for separate mention. To give a simple example, consider the development of $\nabla S\sigma\tau$, which *at once takes* the expanded form $\nabla_1 S\sigma_1\tau + \nabla_1 S\sigma\tau$, the suffix indicating on which vector ∇ is acting. The Italian analysts write this *identity* in the form

$$\text{grad}\,(\sigma \times \tau) = K\frac{d\sigma}{d\rho}\tau + K\frac{d\tau}{d\rho}\sigma$$

where $K\frac{d\sigma}{d\rho}$ means the conjugate of $\frac{d\sigma}{d\rho}$, and indicate how it may be *proved* by appeal to former results. There can be no question as to the merits of the two notations. The quaternion notation is clear and systematic; the other is artificial and cumbrous.

As another example take their expression div$(a\mathbf{u})$ on page 57. This they express in the expanded form

$$I_1\left(a\frac{d\mathbf{u}}{dP}\right) + (\text{grad}\, Ka) \times \mathbf{u},$$

or, using ϕ for a, σ for $\mathbf{u}$, and $d\rho$ for dP,

$$\text{div}\,(\phi\sigma) = I_1\left(\phi\frac{d\sigma}{d\rho}\right) + (\text{grad}\,\phi') \times \sigma.$$

In quaternions

$$\begin{aligned} -\text{div}\,(\phi\sigma) &= S\nabla\phi\sigma \\ &= S\nabla_1\phi\sigma_1 - S\sigma\phi'\nabla, \end{aligned}$$

and we need go no further, for $S\nabla_1\,\phi\sigma_1$ is a perfectly intelligible expression. But it is at once evident from Hamilton's theory given long ago that this is the first invariant of the linear vector function χ, where $\chi\beta = S\beta\nabla_1 \cdot \phi\sigma_1$, or symbolically $S(\quad)\nabla_1 \cdot \phi\sigma_1$, which is their result.

"Non sappiamo esprimere la rot$(a\mathbf{u})$," they say on the same page. Why not? In Quaternions this is $V\nabla\phi\sigma$ which may be expanded in the form

$$V\nabla_1\phi_1\sigma + V\nabla_1\phi\sigma_1$$

in which the suffix shows on which part of the expression the ∇ acts. Each term is perfectly intelligible. In a footnote Burali-Forti and Marcolongo indicate a partial transformation in terms of the vector invariants of the linear vector functions involved. Here again the directness and intelligibility of the quaternion method is abundantly evident.

I have intentionally made this review a comparison of Burali-Forti and Marcolongo's system with Hamilton's; for only in that way can we see what advantage, if any, the new system has. But the Italian analysts take what they regard as a much higher ground. In their preface to the *Omografie Vettoriale* they pride themselves on having presented the homographs or linear vector functions as absolute wholes and not as "tachigraphs" of the coordinates. But this is exactly what Hamilton did with the ϕ, several years before Cayley came forward with the matrix notation. Hamilton's linear vector function is not an array as treated by him. "Further," they say, "it is well to consider linear transformations of vectors in vectorial homographs (of one function of these, namely, the Gradient, we make great use), which cannot be replaced, not even indirectly, by quaternions, when these are employed as absolute wholes, such as they have been given by Hamilton, and not as 'tachigraphs' of Cartesian coordinates." This means that

Burali-Forti and Marcolongo graciously permit Hamilton to use the quaternion as a quantity involving four independent units, but on no account is he to be allowed to consider the properties of *parts* of the quaternion in which one, two, or three only of these units are explicitly contained. The vector part of Hamilton's quaternion is not necessarily a "tachigraph" of Cartesian coordinates. It can be made to represent fully a quantity which in ordinary analysis is represented by the coordinates x, y, z; but that by no means exhausts the possibilities of Hamilton's vector. In truth Burali-Forti and Marcolongo, like many other vector analysts, *appropriate* Hamilton's word "vector," give it a modified and restricted meaning, and then say that Hamilton has no right to use the vector at all. As I think I have shown in the above comparison, the gradient and other linear transformations of vectors in vectorial homographs are part and parcel of the quaternion vector analysis as developed by Hamilton, Tait, McAulay and Joly, and were known to quaternion workers years before Burali-Forti and Marcolongo, by imposing arbitrary limitations upon Hamilton's calculus, ruled it as out of order so that their so-called minimum system might have undisputed sway.

C. G. K.

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THE OLD SCHOOL AND THE NEW.

(*A reminiscence of the three meetings on January* 12*th and* 13*th*, 1910.)

"A mathematician, who 'can't drop things into water,' will soon be as extinct as the Dodo."

"By working in a laboratory mathematicians learn at last really to believe in their own formulae."

Dr. Balance.

"Just drop it into water and you'll very quickly see,
If reckoned and observed results in any way agree."
To this said Dr. Problem, "Into Science I'll not pry,
And I really very much prefer my Mathematics *dry*."
But Doctor Balance then replied with the playful wit of youth,
"Experiment alone decides, if formulae have truth!
And, unless, like Archimedes, you can test a crown of gold,
You'll rapidly become extinct, as the Dodo bird of old!"

H. D. Ellis.

ERRATA.

p. 177, line 5, *for* $\frac{\pi}{3}$ *read* $\frac{2\pi}{3}$.

p. 212, line 6, *delete* 'for.'

line 18, *read* $\lim_{n \to \infty} a^{\frac{1}{n}} = 1$.

lines 22, 23, *interchange* 'ascending' *and* 'descending.'

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